Intermediate Value Theorem (IVT)

Let f be continuous on an interval I.

Then if a, b are in I and z is a point between $f(a)$ and $f(b)$, then there is a point c between a and b with

$$\boxed{f(c) = z}$$

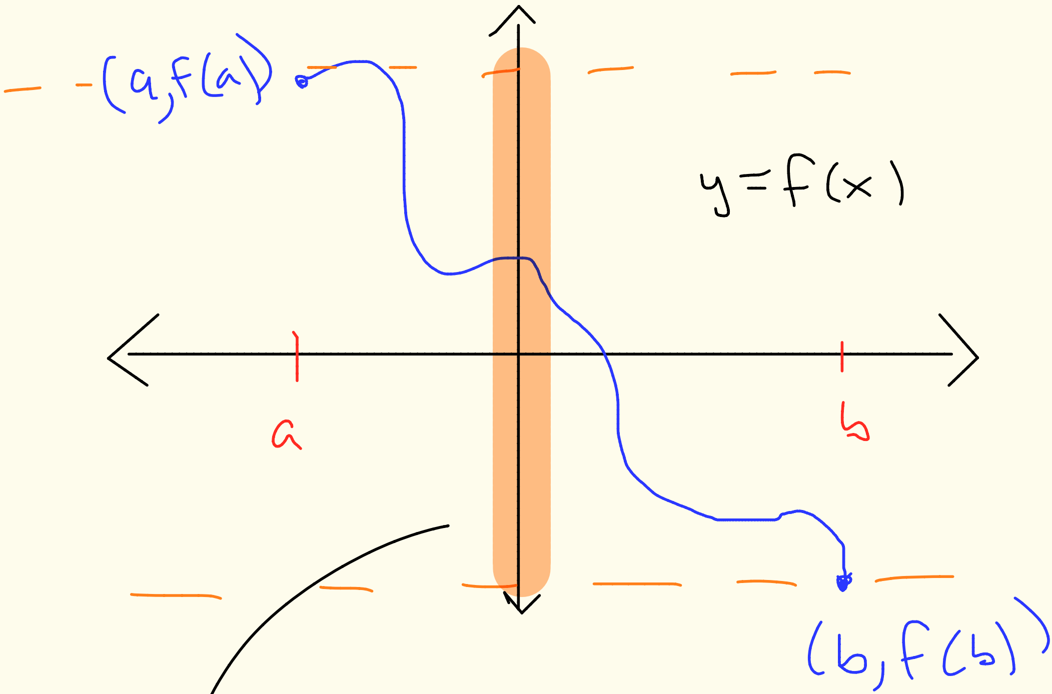
If $f(a) < 0$ and $f(b) > 0$, then the IVT provides a zero of f between a and b but it doesn't tell you how to <u>find</u> the zero

<u>Analogy</u>: A car salesman

tells you that the perfect car for you is out there. When you ask them where, they say they don't know - but they're sure it is out there!

The IVT is the salesman.

Picture (IVT)



f takes on all y-values in the orange strip

Example 1: Show

$$p(x) = -7x^5 - x^4 + 10x^2 - 15x + 1$$

has a zero in $[-1, 1]$.

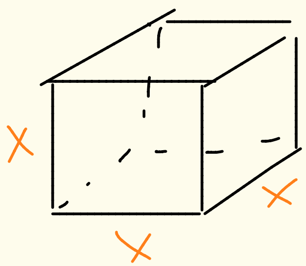
$$p(1) = -7 - 1 + 10 - 15 + 1$$
$$= -12 \boxed{< 0}$$

$$p(-1) = 7 - 1 + 10 + 15 + 1$$
$$= 32 \boxed{> 0}$$

Since p is a polynomial, it is continuous everywhere, so by IVT, p has a zero in $[-1,1]$.

Example 2: Does there exist a cube whose volume is equal to its sidelength plus one?

Picture



Volume of cube

$= x^3$

Sidelength plus one

$= x+1$

We'd need $x^3 = x+1$, so subtracting $x+1$ from both sides, we need

$$\boxed{x^3 - x - 1 = 0}$$

Let $q(x) = x^3 - x - 1$.

$q(2) = 8 - 2 - 1 = 5 \boxed{> 0}$

$q(1) = 1 - 1 - 1 = -1 \boxed{< 0}$

Again since q is a polynomial, q is continuous, so by IVT, q has a zero between 1 and 2, so there is such a cube!

Note: Polynomials are continuous everywhere and rational functions are continuous wherever they are defined, but you must be sure you have continuity to use the IVT!

Example 3: (where things go wrong)

Let $f(x) = \frac{2x-3}{x+1}$.

$f(-2) = 7$ and

$f(0) = -3$, so

does the IVT give you a point c between -2 and 0 with

$f(c) = 2$?

<u>NO</u>

Since f isn't continuous on $[-2,0]$ (vertical asymptote at $x=-1$).

In fact, if

$$2 = f(x) = \frac{2x-3}{x+1}, \text{ then}$$

cross-multiplying,

$$\cancel{2x}+2 = \cancel{2x}-3 \quad \text{and}$$

$$2 = -3 \; !$$